MIDTERM: INTRODUCTION TO ALGEBRAIC GEOMETRY

Date: **5th March 2014**
The total points is **120** and the maximum you can score is **100** points.

A **ring** would mean a **commutative ring with identity**.

(1) (25 pts) When is an affine algebraic set called irreducible? Define radical ideal. Let k be an algebraically closed field and I be an ideal in the polynomial ring $k[X_1, \ldots, X_n]$. Prove or disprove the following:
 a) If I is a radical ideal then the affine algebraic set $Z(I)$ in $\mathbb{A}^n_k$ is irreducible.
 b) If $Z(I)$ is irreducible then I is a radical ideal.
 c) If I is a prime ideal then $Z(I)$ is irreducible.
 d) If $Z(I)$ is irreducible then I is a prime ideal.

(2) (20 points) Let R be a ring and $f : A \to B$ and $g : B \to C$ be R-module homomorphisms. When is the sequence $A \to^f B \to^g C$ called exact? Let F be a (finitely generated) free R-module and

$$0 \to A \to B \to C \to 0$$

be an exact sequence of R-modules. Show that:
 a) $Hom_R(M \oplus N, A) \cong Hom_R(M, A) \oplus Hom_R(N, A)$ for any R-modules M and N.
 b) $0 \to Hom_R(F, A) \to Hom_R(F, B) \to Hom_R(F, C) \to 0$ is exact.

(3) (20 points) Let k be an algebraically closed field. Define affine variety over a field k. Consider the affine algebraic set X in $\mathbb{A}^2_k$ defined by the polynomial $y^n - x \in k[x, y]$ where n is any fixed positive integer. Show that X is a variety. Show that for all n, X is isomorphic to $\mathbb{A}^1_k$.

(4) (20 points) Let A be a ring and B be an A-algebra. Let P be a prime ideal in A and $S = A \setminus P$. Let K be the field of fraction of A/P.
 a) Show that $S^{-1}A/PS^{-1}A$ is isomorphic to K.
 b) Show that $B \otimes_A K$ is isomorphic to $S^{-1}B/PS^{-1}B$.

(5) (20 points) Let A be an integral domain. When is A said to be a normal domain? Let $A \subset B$ be domains and $\alpha \in B$ be integral over A. Let K be the field of fractions of A and assume that $K(\alpha)/K$ is a separable extension. Show that the minimal polynomial of α over K have coefficients in A.

(6) (15 points) Let R be a noetherian local ring and m its maximal ideal and $k = R/m$ the residue field. Show that m/m^2 is a finite dimensional k-vector space and the vector space dimension of m/m^2 is same as the length of the smallest generating set of the ideal m.